A DIRECT PROOF OF MULMULEY'S WEAK $\#\mathcal{P}$ VERSUS $\mathcal{NC}$ RESULT

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Abstract. We present a direct proof of a result Ketan Mulmuley presented at the Institute for Advanced Study during his lecture series "On $\mathcal{P}$ vs $\mathcal{NP}$, Geometric Complexity Theory, and the Riemann Hypothesis" on February 9–11, 2009. The result is intended as a "weak form" of $\#\mathcal{P}$ versus $\mathcal{NC}$ in characteristic zero.

Theorem. *Let $m \geq n \geq 2$, and let Y be an $m \times m$ matrix with variable entries*

$$\begin{cases} x_{i,j} & \text{if } 1 \leq i \leq n \text{ and } 1 \leq j \leq n, \\ y_{i,j} & \text{else.} \end{cases}$$

Call X the top-left $n \times n$ submatrix of Y. Then, there does not exist a polynomial g and complex-valued $m \times m$ matrices B and C so that

$$\operatorname{perm} X = g(\operatorname{tr}(BYC), \ldots, \operatorname{tr}((BYC)^m)).$$

Remark. The proof below carries over to a similar "homogeneous" version stated by Mulmuley.

Proof. We make a series of reductions:

Assuming $y_{i,j} = 0$. We may set all of the y's to zero because none appear in $\operatorname{perm} X$.

Assuming $C = I$. We have that $\operatorname{tr}(BYC) = \operatorname{tr}(CBY)$, and moreover that $\operatorname{tr}((BYC)^i) = \operatorname{tr}((CBY)^i)$, so we may replace B with CB and C with the identity matrix.

Assuming $m = n$. We may assume that $b_{i,j} = 0$ if $j > n$ because those entries do not affect the product BY. Setting D to be the top-left $n \times n$ submatrix of B, we have

$$\operatorname{tr}((BY)^i) = \operatorname{tr}((YB)^i) = \operatorname{tr}((XD)^i) = \operatorname{tr}((DX)^i),$$

so by replacing B with D and Y with X, we have reduced to the case $m = n$.

Assuming $n = 2$. We may pick a vector v with at most two zero entries such that the bottom two entries of Bv are zero. Consider the matrix X consisting of $n-2$ columns of v, and a 2×2 variable submatrix X' in the last two columns in any two rows that include the rows where v is zero. Then BX is block upper-triangular with an $(n-2) \times (n-2)$ submatrix of constants and a 2×2 submatrix $B'X'$. Since X has nontrivial permanent by construction, and constants from $\operatorname{tr}((BX)^i)$ can be folded into g, we can reduce to $n = 2$.

Finish. In the 2×2 case, we may consider polynomials in $\operatorname{tr}(BX)$ and $\det(BX)$ instead of $\operatorname{tr}(BX)$ and $\operatorname{tr}((BX)^2)$ because both pairs generate the same functions. If $X = \left(\begin{smallmatrix} a & a \\ b & b \end{smallmatrix}\right)$, we seek an identity of the form $\operatorname{perm} X = 2ab = g(\operatorname{tr}(BX), \det(BX))$. But $\operatorname{tr}(BX)$ is a linear form in a and b and $\det(BX) = 0$ since X is singular, so this is impossible. ☐

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