AN ANALYSIS OF A WAR-LIKE CARD GAME

BORIS ALEXEEV AND JACOB TSIMERMAN

Abstract. In his book "Mathematical Mind-Benders", Peter Winkler poses the following open problem, originally due to the first author: "[In the game Peer Pressure,] two players are dealt some number of cards, initially *face up*, each card carrying a different integer. In each round, the players simultaneously play a card; the higher card is discarded and the lower card passed to the other player. The player who runs out of cards loses. As the number of cards dealt becomes larger, what is the limiting probability that one of the players will have a winning strategy?"

We show that the answer to this question is zero, as Winkler suspected. Moreover, assume the cards are dealt so that one player receives $r \geq 1$ cards for every one card of the other. Then if $r < \varphi = \frac{1+\sqrt{5}}{2}$, the limiting probability that either player has a winning strategy is still zero, while if $r > \varphi$, it is one.

Introduction

The card game "Peer Pressure", a variant of "War", is played with a deck of n cards, each carrying a distinct integer. The cards are initially dealt randomly to two players, with either exactly $n/2$ cards per player or each card randomly going to one of the players. In each round ("battle") of the game, the players simultaneously play a card. The player holding the higher card wins the round and receives the lower card; however, the higher card is permanently discarded from the game. The player who runs out of cards loses. We assume that both players know the original deck and thus are aware of the contents of both players hands at all times.

Recall that we say a player has a *winning strategy* if she may announce her strategy beforehand and still always win against any strategy from her opponent. A winning strategy may in general be a mixed (probabilistic) strategy, but if one exists, there also exists a pure (deterministic) winning strategy.

In Peer Pressure, if there are four or fewer cards, then one of the players has a winning strategy. In particular, if one player has more cards, then she wins; if the players have an equal number of cards, the player with the highest card wins. However, if there are five cards, there is one position where neither player has a winning strategy: $1, 2, 4$ versus $3, 5$.

Suppose our two players are named Alice and Bob and have a and b cards respectively. We prove the following lemma that helps classify when a player has a winning strategy:

Main Lemma. *Let* $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.61803$ *be the golden ratio, which notably satisfies* $1 + \varphi = \varphi^2$.

- *If Alice has more than* φ *times as many cards as Bob (that is,* $a > \varphi b$*), then Alice has a winning strategy.*
- *If Alice has more than* $1/\varphi$ *times as many cards as Bob (that is,* $b < \varphi a$*) and they are all higher than Bob's, then Alice has a winning strategy.*

We then use this lemma to prove our main result:

Date: October 3, 2011.
2010 *Mathematics Subject Classification.* 91A05.
Key words and phrases. Two-person game, War (card game), winning strategy.

Definition. As is standard in asymptotic analysis, say that a result holds *asymptotically almost surely* (abbreviated *a.a.s.*) if it holds with probability approaching one as the number of cards, n, goes to infinity.

Main Theorem. *In the original game with unbiased dealing, a.a.s. neither player has a winning strategy. Moreover, assume the cards are dealt randomly so that Alice receives $r \geq 1$ cards for every card of Bob. If $r < \varphi$, a.a.s. neither player has a winning strategy, while if $r > \varphi$, a.a.s. Alice has a winning strategy.*

This result determines the limiting probability that one of the players has a winning strategy, an open problem posed by Peter Winkler in his book "Mathematical Mind-Benders", where it is attributed to the first author. [Win07]

Proofs

One interesting quirk in Peer Pressure is that it is not immediately obvious that having better cards is necessarily advantageous. Of course, a better card will be more likely to win in any given round, but the card may also end up in the hands of the opponent, who may then use it to his advantage. We begin by proving that this is not a problem. This lemma is not essential to the later results, but it is instructive.

Definition. We say a collection of cards C is *at least as good as* another collection C' if for all positive integers k, either the kth highest card in C is at least as high as the kth highest card in C' or C' has less than k cards.

Monotonicity Lemma. *Having better cards doesn't hurt. That is, if in a certain position Alice's cards are replaced with better cards and/or Bob's cards are replaced with worse cards, then Alice is no less likely to win. In particular, if Alice had a winning strategy before the replacement, she still does afterward.*

Proof. We prove the result step by step.

Extra cards don't hurt. Suppose Alice receives extra cards, but no other change occurs. Then clearly she is no worse off because she can play the same strategy as before, ignoring her extra cards. If she won before, she still wins. (If she lost before, she now has extra cards that may or may not help in the end.)

Losing cards doesn't help. By symmetry, if Bob has cards taken away, but no other change occurs, he is no better off.

Receiving a card from the opponent doesn't hurt. Suppose Bob gives a card to Alice, but no other change occurs. Then Alice is no worse off because this is equivalent to Alice gaining a card and Bob losing a card.

Slightly improving one card doesn't hurt. Suppose Alice has a card A and Bob has a card B such that $A < B$ before the replacement and $A > B$ after the replacement, but no other change occurs. (In particular, there are no cards of rank between A and B.) Then let Alice play exactly as before *until* one of these cards is played by either player. If both cards are played simultaneously, then Alice is no worse off because she wins a card instead of Bob. Indeed, this is equivalent to Bob winning the battle (as before the replacement), followed by Bob giving Alice a card. If only one card is played and it wins, then it is removed from play and so the relative ranking of A and B doesn't matter anyway. If only one card is played and it loses, then one of the players will end up holding both A and B. Again, the relative ranking doesn't matter because Alice can pretend to switch the cards.

Having better cards doesn't hurt. The general case consists of performing the above modifications one by one. This may be accomplished, for example, by first improving Alice's best card, then her next best, and so on, and afterward, giving Alice extra cards and taking cards away from Bob. □

We may now prove our main lemma.

Main Lemma (again)**.** *Let* $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.61803$ *be the golden ratio, which notably satisfies* $1 + \varphi = \varphi^2$*.*

- *If Alice has more than* φ *times as many cards as Bob (that is,* $a > \varphi b$*), then Alice has a winning strategy.*
- *If Alice has more than* $1/\varphi$ *times as many cards as Bob (that is,* $b < \varphi a$*) and they are all higher than Bob's, then Alice has a winning strategy.*

Proof. We prove the result by induction on the total number of cards, $a + b$. Note that both results are certainly true when $a = 0$ or $b = 0$.

Alice has many cards. Suppose that $a > \varphi b > 0$. In order to win, Alice plays her current a cards in order from low to high, ignoring Bob's actions (notably not reusing any cards Bob gives back to her). Alice loses at most b of these battles, so afterwards she has at least $a - b$ cards. The total number of cards remaining is b and so Bob has at most $2b - a$ cards. The result holds by induction because $(a-b) > \varphi(2b-a)$.

Alice has enough high cards. Suppose that $\varphi a > b > 0$ and all of Alice's cards are higher than Bob's. In order to win, Alice plays each of her a cards once. She will win every battle, so afterward she will still have a cards, while Bob will have $b - a$ cards. The result holds by induction because $a > \varphi(b - a)$. □

Remark. Note that once Alice finds herself in either case of the Main Lemma, she may win by repeatedly playing her cards from low to high without regard to Bob's actions.

Armed with the Main Lemma, we prove the Main Theorem. Recall that a result holds *asymptotically almost surely (a.a.s.)* if it holds with probability approaching one as the number of cards, n, goes to infinity.

Main Theorem (again)**.** *In the original game with unbiased dealing, a.a.s. neither player has a winning strategy. Moreover, assume the cards are dealt randomly so that Alice receives* $r \geq 1$ *cards for every card of Bob. If* $r < \varphi$*, a.a.s. neither player has a winning strategy, while if* $r > \varphi$*, a.a.s. Alice has a winning strategy.*

Proof. First of all, because only the relative ordering of the cards matters, we assume the cards are numbered 1 to n. Also, recall that if Alice has a mixed (probabilistic) winning strategy, then she also has a pure (deterministic) winning strategy.

We begin with the weaker result with unbiased dealing. Divide the cards into five equally-sized intervals: $C_1 = (0, n/5]$, $C_2 = (n/5, 2n/5]$, $\ldots$, $C_5 = (4n/5, n]$. Suppose Alice reveals her pure strategy in advance to Bob. We will show how Bob can use these intervals to defeat it. Specifically, for $1 \leq i < 5$, he will use his cards in C_{i+1} to defeat Alice's cards in C_i. Finally, he will use his leftover cards to defeat Alice's C_5. See the Figure for a visual explanation.

We expect each player to receive half of the cards in each interval, so by the law of large numbers, Bob will a.a.s. receive at least $.099n$ cards from each interval; Alice will thus receive at most the remainder, $.101n$. Now suppose that only Bob's cards in C_2 and Alice's cards in C_1 are under consideration. By the Main Lemma, Bob may use $.063n > .101n/\varphi$ cards from C_2 to defeat Alice's cards in C_1 (note that all of the cards in C_2 are higher than those in C_1), leaving at least $(0.099 - 0.063)n = 0.036n$ unused cards left over. If he does similarly for his C_3 through C_5, Bob will have at least $4(.036n) + .099n = .243n$ cards left over. Again by the Main Lemma, Bob may use these cards to defeat Alice's C_5 because $.243/.101 > \varphi$.

In the previous paragraph, we pretended that Bob may consider the game as the sum of five independent games. However, this is justified because Alice has revealed her pure strategy in advance. Because Bob

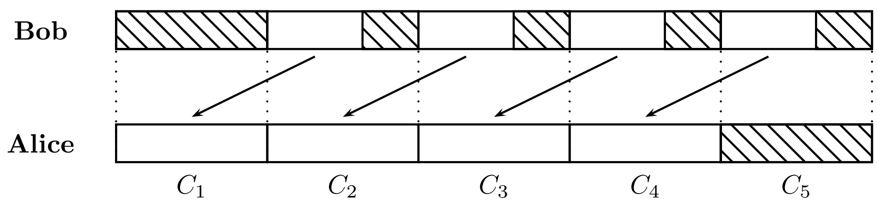

FIGURE. The unshaded intervals illustrate how Bob uses slightly more than a $1/\varphi$-proportion of his cards in C_{i+1} to defeat Alice's cards in C_i. Bob's leftover cards in all of his C_i, represented by shaded intervals, are sufficient in number to overwhelm Alice's cards in C_5.

knows where Alice will play, he may use the appropriate cards to defeat her. Note that Bob may choose *beforehand* which cards are allocated where, so it does not matter in what order Alice plays; in particular, Bob can choose his "leftover cards" beforehand, as all that matters is their number. Therefore a.a.s., Alice has no winning strategy and by symmetry, neither does Bob.

Now we prove the stronger result with dealing biased towards Alice. If $r > \varphi$, this is easy. Alice will a.a.s. have more than φ times as many cards as Bob and thus win by the Main Lemma.

Now suppose that $r < \varphi$ is fixed. We follow the same approach as before. Divide the cards into k equally-sized intervals $C_i = \left(\frac{i-1}{k}n, \frac{i}{k}n\right]$, where k will be chosen later to depend only on r and not on n. In each interval, we expect players to receive cards in an $r : 1$ proportion. By the law of large numbers, for any constant $\delta > 0$, Bob a.a.s. receives at least $1 - \delta$ of the number of cards he expects in *each* of the intervals. (Note that we crucially use here that k does not depend on n.) By the Main Lemma, we may choose δ so small (but independent of k) that Bob may use his cards in C_{i+1} to defeat Alice's cards in C_i for all $1 \leq i < k$ and still have a positive proportion of his cards left over in each interval.

Now choose k so large (but independent of n) that Bob's remaining cards in C_1 and his leftover cards from all of the other C_i are more than φ times the number of Alice's cards in C_k. This is possible because we insured that each C_i has at least a fixed positive proportion of cards left over, so Bob may overwhelm Alice with his extra cards. Now, as before, Bob's leftover cards defeat Alice's C_k by the Main Lemma. (Again, Bob's cards may be allocated before any actual play.) Finally, if Alice reveals her strategy in advance, Bob may combine his strategies on all of the intervals C_i to defeat her.

Therefore, if $r < \varphi$, Alice a.a.s. does not have a winning strategy. Bob a.a.s. doesn't have a winning strategy either, because his cards are even worse than in the unbiased case. □

FURTHER DIRECTIONS

The results in this paper may be continued in a few natural directions.

For example, by using techniques similar to those presented above, Jacob Fox has determined the threshold for the number of battles a player can guarantee winning in the unbiased model. [Fox08] In particular, if $f(n)$ is a function that grows slower than $\sqrt{n}$ (using Landau's asymptotic notation, $o(\sqrt{n})$), then a.a.s. both of the players may guarantee winning at least $f(n)$ of the battles. However, if $f(n)$ is a function that grows faster than $\sqrt{n}$ (using Landau's asymptotic notation, $\omega(\sqrt{n})$), then a.a.s. neither player may guarantee winning at least $f(n)$ of the battles. The threshold $\sqrt{n}$ comes from the central limit theorem.

In another direction, note that the Main Lemma classifies some of the hands where Alice has a winning strategy, and it can also be used to classify some hands where neither player has a winning strategy (as in the Main Theorem). We leave as an open problem whether or not one may prove stronger results about winning strategies:

Problem. Classify all situations when a given player has a winning strategy.

Acknowledgments

The authors wish to thank Dan Cranston and Jacob Fox for helpful discussions.

References

[Fox08] Jacob Fox, personal communication, November 2008.

[Win07] Peter Winkler, *Mathematical mind-benders*, ch. 11, p. 134, A K Peters Ltd., Wellesley, MA, 2007.

(B. Alexeev) Mathematics Department, Princeton University, Fine Hall, Washington Road, Princeton, NJ 08544

(J. Tsimerman) Department of Mathematics, Harvard University, One Oxford Street, Cambridge, MA 02138

E-mail address: balexeev@math.princeton.edu, jacobt@math.harvard.edu