On the complexity of Mumford-Shah type regularization, viewed as a relaxed sparsity constraint

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Abstract

We show that inverse problems with a *truncated quadratic regularization* are NP-hard in general to solve, or even approximate up to an additive error. This stands in contrast to the case corresponding to a finite-dimensional approximation to the Mumford-Shah functional, where the operator involved is the identity and for which polynomial-time solutions are known. Consequently, we confirm the infeasibility of any natural extension of the Mumford-Shah functional to general inverse problems. A connection between truncated quadratic minimization and sparsity-constrained minimization is also discussed.

inverse problems, Mumford-Shah functional, truncated quadratic regularization, sparse recovery, NP-hard, thresholding, SUBSET-SUM

1 Introduction

Consider a discrete signal $x \in \mathbb{R}^N$ sampled from a piecewise smooth signal and revealed through measurements $y = Ax + e$, where $e \in \mathbb{R}^m$ is observation noise and $A : \mathbb{R}^N \mapsto \mathbb{R}^m$ is a known linear operator identified with an $m \times N$ real matrix (representing, for instance, a blurring or partial obscuring of the data). Consider the *truncated quadratic minimization problem*,

$$\hat{x} = \operatorname*{argmin}_{x \in \mathbb{R}^N} \mathcal{J}(x),$$
$$\mathcal{J}(x) = \|Ax - y\|_2^2 + \sum_{j=1}^{N-1} Q(x_{j+1} - x_j), \tag{1}$$

with truncated quadratic penalty term $Q(u) = \alpha \min\{u^2, \beta\}$ parametrized by $\alpha, \beta > 0$. Since its introduction in 1984 by Geman and Geman in the context of image restoration [2, 6, 8], this problem has been the subject of considerable theoretical and practical interest, finding applications ranging from visual analysis to crack detection in fracture mechanics [9, 10]. The choice of regularization is motivated as follows: Q desires to smooth small differences $|x_{j+1} - x_j| \leq \sqrt{\beta}$ where it acts

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quadratically, but suspends smoothing over larger differences.

From a statistical point of view, the quadratic data-fidelity term $\|Ax - y\|_2^2$ can be viewed as a log-likelihood of the data under the hypothesis that e is Gaussian random noise, while the truncated quadratic regularization term corresponds to the energy of a piecewise Gaussian Markov random field [1, 6, 7].

The truncated quadratic minimization problem is non-smooth and highly non-convex. However, several characterizations of the minimizers have been unveiled [5, 8]. It is known for instance that minimizers exist and satisfy a "gap" property [5, 8]: the magnitude of successive differences of such solutions are either smaller than a first threshold or larger than a second, *strictly* larger threshold. These thresholds are independent of the observed data $y = Ax + e$ and depend only on the regularization parameters α and β. This dependence is explicit, so that a priori information about the thresholds can be incorporated into choice of regularization parameters.

When A is the $N \times N$ identity matrix, the truncated quadratic objective function can be viewed as a discretization of the *Mumford-Shah functional*[1], which motivated the variational approach for edge detection and image segmentation with its introduction in 1988. When A is the identity matrix as such, the truncated quadratic minimization problem can be solved in polynomial-time using dynamic programming [4]. However, for general $m \times N$ matrices A, existing algorithms for minimizing the functional (1) guarantee convergence to *local* minimizers at best [5].

In this paper, we show that the truncated quadratic minimization problem is NP-hard in general, certifying that the present convergence guarantees are the best one could hope for. Consequently, the Mumford-Shah functional (2) cannot be tractably extended to general inverse problems.

2 Truncated quadratic minimization reformulated

It will be helpful to recast the truncated quadratic minimization problem in terms of the discrete differences $u_j = x_{j+1} - x_j$, effectively decoupling the action of the regularization term Q. We may express this change of variables in matrix notation

[1] In [4], the minimizers $\hat{x} = \hat{x}_{(N)}$ of the N-dimensional truncated quadratic minimization problem with $A = I$, parameters $(\alpha_{(N)}, \beta_{(N)}) = (N^2\alpha, N\beta)$, and $y_{(N)} = y_{(j/N)}$ identified with discrete samples from a continuous function $y \in L^\infty[0,1]$, were shown to converge to the minimizer of the Mumford-Shah functional,

$$\begin{aligned} \hat{x} &= \underset{x \in SBV[0,1]}{\operatorname{argmin}} \; \mathcal{F}(x), \\ \mathcal{F}(x) &= \int_{[0,1]\setminus S_x} \Big((x-y)^2 + \alpha \|\nabla x\|_2^2 \Big) dx + \alpha\beta \, |S_x| \, , \end{aligned}$$

over the space SBV of bounded variation functions on $[0,1]$ with vanishing Cantor part. Note that SBV functions have a well-defined *discontinuity set* S_x of finite cardinality $|S_x|$; see [5] for more details.

as $u = Dx$, with $D : \mathbb{R}^N \to \mathbb{R}^{N-1}$ the *discrete difference* matrix,

$$D = \begin{pmatrix} -1 & 1 & 0 & \dots & \dots & 0 \\ 0 & -1 & 1 & 0 & \dots & 0 \\ \dots & & & & & \\ 0 & 0 & \dots & \dots & -1 & 1 \end{pmatrix}.$$

The null space of D, which we denote in the following by $\mathcal{N}(\mathcal{D})$, is simply the one-dimensional subspace of constant vectors in $\mathbb{R}^N$. The orthogonal projection of a vector $x \in \mathbb{R}^N$ onto this subspace is the constant vector c whose entries coincide with the mean value $\frac{1}{N}\sum_{j=1}^{N} x_j$ of x, while its projection onto the orthogonal complement of $\mathcal{N}(\mathcal{D})$ is given by the least squares solution $D^\dagger D x$, where $D^\dagger$ is the pseudo-inverse matrix of D in the Moore-Penrose sense. These observations yield the orthogonal decomposition $x = D^\dagger D x + c$, or, incorporating the substitution $u = Dx$, the decomposition $x = D^\dagger u + c$.

Minimization of $\mathcal{J}$, recast in terms of the variables u and c, becomes

$$(\hat{c}, \hat{u}) = \underset{c \in N(D),\, u \in \mathbb{R}^{N-1}}{\operatorname{argmin}} \mathcal{J}(c, u),$$

$$\mathcal{J}(c, u) = \left\| A D^\dagger u + Ac - y \right\|_2^2 + \sum_{j=1}^{N-1} Q(u_j), \tag{2}$$

where the primal minimizer $\hat{x}$ and (c, u)-minimizer $(\hat{c}, \hat{u})$ are interchangeable according to $\hat{x} = D^\dagger \hat{u} + \hat{c}$.

If the null space of A contains the constant vectors, such as if $A = TD$ for an $m \times (N-1)$ matrix T, the minimization problem (2) reduces to a function of u only,

$$\hat{u} = \underset{u \in \mathbb{R}^{N-1}}{\operatorname{argmin}} \left\| A D^\dagger u - y \right\|_2^2 + \sum_{j=1}^{N-1} Q(u_j).$$

Making the substitution $A = TD$ and using that $DD^\dagger = I$, we see in particular that any optimization problem of the form $\hat{u} = \operatorname{argmin}_{u \in \mathbb{R}^{N-1}} \|Tu - y\|_2^2 + \sum_{j=1}^{N-1} Q(u_j)$ can be identified with an instance of a truncated quadratic minimization problem (1). To summarize,

Lemma 1. *Let $T : \mathbb{R}^{N-1} \mapsto \mathbb{R}^m$ be a linear operator identified with a matrix of $\mathbb{R}^{m \times (N-1)}$. The minimization problem*

$$\hat{u} = \underset{u \in \mathbb{R}^{N-1}}{\operatorname{argmin}} \|Tu - y\|_2^2 + \sum_{j=1}^{N-1} Q(u_j) \tag{3}$$

corresponds to a truncated quadratic minimization problem

$$\hat{x} = \underset{x \in \mathbb{R}^N}{\operatorname{argmin}} \|TDx - y\|_2^2 + \sum_{j=1}^{N-1} Q(x_{j+1} - x_j) \tag{4}$$

in the sense that $\hat{x} = D^\dagger \hat{u}$ is a minimizer for (4) if $\hat{u}$ minimizes (3), while $\hat{u} = D\hat{x}$ is a minimizer for (3) if $\hat{x}$ is a minimizer for (4).

3 Reduction to SUBSET-SUM

Recall that the complexity class NP consists of all problems whose solution can be verified in polynomial time given a certificate for the answer. For example, the problem SUBSET-SUM is to determine, given nonzero integers $a_1, \ldots, a_k$ and C, whether or not there exists a subset S of $\{1, \ldots, k\}$ such that $\sum_{i \in S} a_i = C$. This problem is in NP because given any particular subset S, we can easily check whether or not its corresponding sum is zero.

Further recall that a *polynomial-time many-one reduction* from a problem A to a problem B is an algorithm that transforms an instance of A to an instance of B with the same answer in time polynomial with respect to the number of bits used to represent the instance of A. Intuitively, this captures the notion that A is no harder than B, up to polynomial factors, and accordingly one may write $A \leq B$. Finally, a problem B is called NP-hard if every problem in NP is reducible to B. (Note that $A \leq B$ and $B \leq C$ imply $A \leq C$, so if an NP-hard problem B reduces to a problem C, then C is NP-hard as well.) NP-hard problems can not be solved in polynomial time unless P=NP.

In order to prove our NP-hardness result, we show that the known NP-hard problem SUBSET-SUM admits a polynomial-time reduction to an instance of the truncated quadratic minimization problem. Moreover, we show that any algorithm that could efficiently *approximate* this minimum (to within an additive error) could solve SUBSET-SUM efficiently as well; that is, the search for even an approximate solution to a truncated quadratic minimization problem is NP-hard as well.

Theorem 2. *Let $a_1, \ldots, a_k$ and C be given nonzero integers. Then there exists a subset S of $\{1, \ldots, k\}$ such that $\sum_{i \in S} a_i = C$ if and only if $\min_{x \in \mathbb{R}^{2k}} f(x) \leq k$, where*

$$f(x) = \left(C - \sum_{i=1}^{k} a_i x_i\right)^2 + P \cdot \sum_{i=1}^{k} (1 - x_i - x_{i+k})^2 + \sum_{i=1}^{2k} \min\left(1, \frac{x_i^2}{\varepsilon^2}\right),$$

with $0 < \varepsilon \leq \frac{1}{4(\sum_i |a_i|)}$ and $P \geq \frac{2k}{\varepsilon^2}$. Moreover, this minimum is never strictly between k and $k + \frac{1}{4}$.

Call a subset $S \subseteq \{1, \ldots, k\}$ *good* if $\sum_{i \in S} a_i = C$. If a good subset S exists, we may set

$$x_i = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{if } i \notin S, \end{cases} \qquad \text{and} \qquad x_{i+k} = \begin{cases} 0 & \text{if } i \in S, \\ 1 & \text{if } i \notin S, \end{cases}$$

for $1 \leq i \leq k$. Then $f(x) = k$ because the first and second terms vanish, and there are exactly k nonzero x_is. Therefore, if a good subset exists, the minimum is at most k.

Suppose no good subset exists, yet there exists x such that $f(x) < k + \frac{1}{4}$. Consider the k pairs of coordinates x_i, x_{i+k} for $1 \leq i \leq k$. If both $|x_i|$ and $|x_{i+k}|$

were less than ε, a single summand in the second term already exceeds $2k$, as $P \cdot (1 - x_i - x_{i+k})^2 \geq P \cdot (1 - 2\varepsilon)^2 \geq 2k$. Therefore, at least one of $|x_i|$ and $|x_{i+k}|$ exceeds ε and the third term is already at least k. If more than one of $|x_i|$ and $|x_{i+k}|$ exceeded ε, then the third term would be at least $k+1$, so exactly one of the coordinates in each pair exceeds ε in absolute value. If $|x_i| \leq \varepsilon$, then $|x_{i+k} - 1| \leq 2\varepsilon$ as otherwise $P \cdot (1 - x_i - x_{i+k})^2 \geq P \cdot (\varepsilon)^2 \geq 2k$; the symmetric holds if $|x_{i+k}| \leq \varepsilon$, so all of the x_i are within 2ε of either 0 or 1.

Let $\overline{x}_i$ be the closer of 0 and 1 to x_i. Then because no good subset exists and C and the a_i are integers, $\left|C - \sum_{i=1}^k a_i \overline{x}_i\right| \geq 1$. It follows that the first term $\left(C - \sum_{i=1}^k a_i x_i\right)^2 \geq (1 - \sum 2\varepsilon a_i)^2 \geq (\frac{1}{2})^2 = \frac{1}{4}$. But the third term was already at least k, so this is a contradiction. Therefore, if no good subset exists, we must have $\min_x f(x) \geq k + \frac{1}{4}$.

Corollary 3. *Solving the truncated quadratic regularization problem, even to within an additive error, is NP-hard.*

In light of Lemma 1, minimization of the function f is a truncated quadratic minimization problem, with $m = k+1$ and $N = 2k$. Therefore, we have reduced the known NP-hard problem SUBSET-SUM to a truncated quadratic minimization problem. It remains to verify that this reduction is polynomial-time. To see this, note that Theorem 2 ensures that the minimum of f is either at most k or at least $k + \frac{1}{4}$; thus, we only need to approximate each of the polynomially-many entries in the matrices and vectors in f to within a number of bits that is polynomial compared to the number of bits needed to represent $\sum_i |a_i|$.

4 Connection to sparse recovery

The only properties of the quadratic regularization term $\min\{1, x_i^2/\varepsilon^2\}$ needed for Theorem 2 were that it be bounded between 0 and 1, equal to 0 if $x_i = 0$, and equal to 1 if $|x_i| \geq \varepsilon$. Indeed, Theorem 2 holds for *any* regularization term satisfying these properties; for example, one could consider hard thresholding,

$$|x|_0 = \begin{cases} 0 & x = 0, \\ 1 & x \neq 0, \end{cases}$$

which generates the ℓ_0 "counting norm" $\|x\|_0 = \sum_{j=1}^N |x_j|_0$. We then reprove the known result that the ℓ_0-regularized optimization problem,

$$\hat{u} = \operatorname*{argmin}_{u \in \mathbb{R}^N} \|Tu - y\|_2^2 + \gamma \|u\|_0 ,$$

is NP-hard in general. This functional is of considerable interest in the emerging area of sparse recovery, as it is guaranteed to produce sparse solutions for sufficiently large γ and over a certain class of matrices [2]. In this light, the truncated quadratic

[2]See [3] for matrix constructions that admit polynomial-time recovery algorithms for the ℓ_0-regularized optimization problem. All constructions at present involve an element of randomness, and a complete characterization of such matrices forms the core of the area known as *compressed sensing.*

minimization problem may be interpreted as a relaxation of the ℓ_0-regularized optimization problem, and our main result as showing that even such *relaxations* of the ℓ_0-regularized functional are NP-hard.

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