

# EQUATIONS RESOLVING A CONJECTURE OF RADO ON PARTITION REGULARITY

BORIS ALEXEEV AND JACOB TSIMERMAN

ABSTRACT. A linear equation  $L$  is called  $k$ -regular if every  $k$ -coloring of the positive integers contains a monochromatic solution to  $L$ . Richard Rado conjectured that for every positive integer  $k$ , there exists a linear equation that is  $(k - 1)$ -regular but not  $k$ -regular. We prove this conjecture by showing that the equation  $\sum_{i=1}^{k-1} \frac{2^i}{2^i - 1} x_i = \left(-1 + \sum_{i=1}^{k-1} \frac{2^i}{2^i - 1}\right) x_0$  has this property.

This conjecture is part of problem E14 in Richard K. Guy's book "Unsolved problems in number theory", where it is attributed to Rado's 1933 thesis, "Studien zur Kombinatorik".

In 1916, Schur [Sch16] proved that in any coloring of the positive integers with finitely many colors, there is a monochromatic solution to  $x + y = z$ . In 1927, van der Waerden [vdW27] proved his celebrated theorem that every finite coloring of the positive integers contains arbitrarily long monochromatic arithmetic progressions. In his famous 1933 thesis, Richard Rado [Rad33] generalized these results by classifying the systems of linear equations with monochromatic solutions in every finite coloring. His thesis also contained conjectures regarding equations that *do* have a finite coloring with no monochromatic solutions.

**Definition.** A linear equation  $L$  is  $k$ -regular if every  $k$ -coloring of the positive integers contains a monochromatic solution to  $L$ .

*Remark.* Some authors require that the values of the variables be distinct in solutions to  $L$ . We follow Rado and Guy in not including this extra condition.

**Conjecture** (Rado, [Rad33] via [Guy04]<sup>1</sup>). For every positive integer  $k$ , there exists a linear equation that is  $(k - 1)$ -regular but not  $k$ -regular. In other words,  $k$  is the least number of colors in a coloring of the positive integers with no monochromatic solution to  $L$ .

Fox and Radoičić [FR05] conjectured that the family of equations  $M_k$  given by  $\sum_{i=0}^{k-2} 2^i x_i = 2^{k-1} x_{k-1}$  has this property. We prove Rado's conjecture by using the related family of equations,

$$\sum_{i=1}^{k-1} \frac{2^i}{2^i - 1} x_i = \left(-1 + \sum_{i=1}^{k-1} \frac{2^i}{2^i - 1}\right) x_0,$$

which we denote by  $L_k$ .

**Theorem 1.** *The equation  $L_k$  is  $(k - 1)$ -regular but not  $k$ -regular.*

*Remark.* This result and its proof carry over to colorings of the nonzero rationals.

*Proof.* We first use the power of 2-adic valuations to prove that there exists a  $k$ -coloring with no monochromatic solutions to  $L_k$ . The idea of using valuations was proposed by Fox and Radoičić for the equation  $M_k$ ;

*Date:* June 3, 2009.

2000 *Mathematics Subject Classification.* 05D10.

*Key words and phrases.* Colorings, partition regularity, Ramsey theory.

<sup>1</sup>These authors could not verify whether the conjecture is present in Rado's thesis.

later, Alexeev, Fox, and Graham [AFG07] proved that these colorings were actually minimal if  $k \leq 7$ , but only conjectured the result in general.

If  $r$  is a nonzero rational number, let  $\text{ord}_2(r)$  denote the 2-adic valuation of  $r$ , that is, the unique integer  $m$  such that  $r = 2^m \frac{a}{b}$  for odd integers  $a$  and  $b$ ; also, let  $\text{ord}_2(0) = \infty$ . Recall that  $\text{ord}_2$  satisfies the following two properties:

- (1)  $\text{ord}_2(xy) = \text{ord}_2(x) + \text{ord}_2(y)$ ,
- (2)  $\text{ord}_2(x + y) \geq \min(\text{ord}_2(x), \text{ord}_2(y))$ , with equality if  $\text{ord}_2(x)$  does not equal  $\text{ord}_2(y)$ .

The latter is the (non-Archimedean) 2-adic triangle inequality.

Note that the 2-adic valuation of the coefficient of  $x_i$  in  $L_k$  is  $i$ . This is immediate if  $i > 0$  while in the case of  $i = 0$ , this follows from the 2-adic triangle inequality because  $-1$  has valuation 0 while the rest of the terms in the summation have positive valuation. We claim this implies that the  $k$ -coloring  $C_k(r) = \text{ord}_2(r) \bmod k$  has no monochromatic solutions to  $L_k$ .

Assume to the contrary that  $C_k$  admits a monochromatic solution to  $L_k$ . Then the terms of  $L_k$  would be a collection of numbers with distinct 2-adic valuations that sum to zero, which is impossible by the 2-adic triangle inequality.

We now show there is no coloring with fewer than  $k$  colors. Indeed, in any coloring with no monochromatic solutions to  $L_k$ , the color of  $x$  and  $2^j x$  must be different if  $0 < j < k$ . (In the referenced literature, the number  $2^j$  is thus said to be a *forbidden ratio*.) If there were an  $x$  so that  $x$  and  $2^j x$  were the same color, then

$$x_i = \begin{cases} x & \text{if } i = j, \\ 2^j x & \text{if } i \neq j, \end{cases}$$

would be a monochromatic solution to  $L_k$ ; all but the  $j^{\text{th}}$  terms cancel, leaving  $\frac{2^j}{2^j-1}x = \left(-1 + \frac{2^j}{2^j-1}\right)2^j x$ . This implies that the  $k$  numbers  $1, 2, 4, \dots, 2^{k-1}$  are colored with distinct colors.  $\square$

#### ACKNOWLEDGMENTS

The authors wish to thank Owen Biesel for helpful comments on the exposition.

#### REFERENCES

- [AFG07] Boris Alexeev, Jacob Fox, and Ron Graham, *On minimal colorings without monochromatic solutions to a linear equation*, Combinatorial number theory, de Gruyter, Berlin, 2007, pp. 1–22.
- [FR05] Jacob Fox and Rados Radoičić, *The axiom of choice and the degree of regularity of equations over the reals*, preprint, Dec 2005.
- [Guy04] Richard K. Guy, *Unsolved problems in number theory*, third ed., Problem Books in Mathematics, ch. E14, p. 328, Springer-Verlag, New York, 2004.
- [Rad33] Richard Rado, *Studien zur Kombinatorik*, Math. Z. **36** (1933), no. 1, 424–470.
- [Sch16] Issai Schur, *Über die Kongruenz  $x^m + y^m \equiv z^m \pmod{p}$* , Jahresber Deutsch. Math. Verein. **25** (1916), 114–117.
- [vdW27] Bartel Leendert van der Waerden, *Beweis einer Baudet’schen Vermutung*, Nieuw Arch. Wiskunde **15** (1927), 212–216.

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, FINE HALL, WASHINGTON ROAD, PRINCETON, NJ 08544-1000  
*E-mail address:* `balexeev@math.princeton.edu`

*E-mail address:* `jtsimerm@math.princeton.edu`